MAX-PLANCK-INSTITUT FÜR DEMOGRAFISCHE FORSCHUNG

MAX PLANCK INSTITUTE FOR DEMOGRAPHIC RESEARCH

Lecture: General Equilibrium

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Workshop Objective

This workshop is intended for people interested in modeling general equilibrium overlapping generations models of a closed economy. These models are used for a wide variety of economic problems such as monetary and fiscal policies, economic growth, and the optimal age of retirement, among many others. In this course I aim to give the basic knowledge to model a general OLG model as well as the intuition to improve it. We will assume that individuals can live N periods, face mortality risk, have perfect foresight, and give and receive transfers along their life cycles. We will simulate the economy using the discrete version of the Bellman equation.

1 Introduction

A computable general equilibrium model aims to derive the optimal allocation process of the households, maximize the profits of firms, and satisfy the objective function(s) of any other economic agent(s) that we are taken into account. In this course I will present a technique that helps to model economies with realistic demography (it can handle single-age groups) for several centuries (at least 600 years). This technique relies on the widely-used Bellman equation.

2 Assumptions

- Closed economy: Productive capital (K_t) = Household asset holdings $\{a_{t,x}\}_{x=0}^{\Omega-1}$

$$K_t = \sum_{x=0}^{\Omega-1} a_{t,x} N_{t,x} \tag{1}$$

- Constant returns to scale on capital and labor (Cobb-Douglas production function)

$$F(K_t, A_t L_t) = (K_t)^{\alpha}(A_t L_t)^{1-\alpha} \Rightarrow \begin{cases} r_t = \alpha \left(\frac{K_t}{A_t L_t}\right)^{\alpha-1} - \delta \\ \\ w_t = \alpha \left(\frac{K_t}{A_t L_t}\right)^{\alpha} \end{cases} \Rightarrow w_t = \alpha \left(\frac{r_t + \delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \tag{2}$$

- There is no annuity market, as a consequence individuals cannot borrow money during their lifespan $\{a_{t+x,x} \geq 0\}_{x=1}^{\Omega-1}$ for all t.

- Individuals are rational, have perfect foresight, and face mortality risk: (use the lifecycle theory of savings)

$$Max_{\{c_{t+x,x}\geq 0\}_{x=0}^{\Omega-1}, \{a_{t+x,x}\geq 0\}_{x=1}^{\Omega-1}} \sum_{x=0}^{\Omega-1} \beta^x \left(\prod_{s=0}^{i} p_{t+s,s}\right) u(c_{t+x,x}) \quad \text{subject to:} \tag{3}$$

$$c_{t+x,x} = (1 + r_{t+x})a_{t+x,x} - a_{t+x+1,x+1} + A_{t+x}\epsilon_x w_{t+x} + \tau_{t+x,x} > 0, \text{ for all } x. \tag{4}$$

Table 1: Parameter values

Name	Parameters	Value
Household		
Age at first entrance into the labor market	T_w	21
Retirement age	T_r	65
Maximum age	Ω	101
Subjective discount factor	β	0.99

3 Household problem: Bellman Equations

3.1 Theoretical approach

The aim of the head of the household of age $x \in \mathcal{X} = \mathcal{L} \bigcup \mathcal{R} = \{T_w, \ldots, \Omega - 1\}$ in year $t \in \mathcal{T} = \{t_0, t_0 + 1, \ldots, T\}$ is to maximize her expected utility by choosing the optimal household consumption in period t (λc) and asset holdings in period $t+1$ (a'). The Bellman equation for the head of the household reads as

$$v(a) = \max_{c,a'} \left\{ \lambda u(c) + \beta \cdot p \cdot v(a') \right\} \tag{5}$$

subject to

$$a' = \begin{cases} (1+r)(a+h) + y_l + \tau^f_{int} + \tau^p - \lambda c & \text{if } x \in \mathcal{L}, \\ (1+r)(a+h) + \tau^f_{int} + \tau^p - \lambda c & \text{if } x \in \mathcal{R}, \end{cases} \tag{6}$$

and the boundary conditions

$$c > 0, a \geq 0, \text{ with } a_{\cdot,T_w} = a_{\cdot,\Omega} = 0. \tag{7}$$

where r is the (real) interest rate, $\lambda \geq 1$ is the number of equivalent adult consumers in the household, $\beta \in (0,1)$ is the subjective discount factor, $p \in [0,1)$ is the probability of surviving to next period conditional on being alive at the beginning of the period, h is the bequest received, τ^f_{int} is net inter-household transfers received, τ^p is net public transfers received, y_l is the gross salary, and a denotes asset holdings.

Note that there is one control variable and one state variable to be maximized $\{c, a'\}$. We are going to transform the household problem by rearranging terms in equation (6) so that we only have to deal with variable a. To do so, let define $G(a, a')$ as the correspondence of total amount of consumption goods attainable for any combination of asset holdings in the present and in the future; that is,

$$c = G(a, a') = \begin{cases} \frac{1}{\lambda}\left(\underbrace{ra - (a' - a)}_{\text{ABR}} + \tau^f_{int} + \tau^p + \underbrace{(1+r)h}_{\text{Bequests}} + y_l \right) & \text{if } x \in \mathcal{L}, \\ \\ \frac{1}{\lambda}\Big(ra - (a' - a) + \tau^f_{int} + \tau^p + (1+r)h \Big) & \text{if } x \in \mathcal{R}. \end{cases} \tag{8}$$

Let define the set $C \subset \mathbf{R}_+ \times \mathbf{R}_+$ as the region of pairs $(a, a') \in \mathbf{R}^2_+$ where consumption is nonnegative; that is, $C = \{(a, a') \in \mathbf{R}^2_+ : G(a, a') \geq 0, \text{ for any given } x\}$. It is easy to prove that C is a convex set. Now, using C let us rewrite the Bellman equation as

$$v(a) = \max_{\text{given } a, a' \geq 0} \left\{ \lambda u(C) + \beta \cdot p \cdot v(a') \right\}. \tag{9}$$

First-order-conditions: analytical approach

Differentiating the right-hand-side of Equation (9) by a' gives

$$\lambda u'(C) \frac{dc}{da'} + \beta \cdot p \cdot v'(a') + \theta = 0 \Rightarrow u'(C) = \beta \cdot p \cdot v'(a') + \theta. \tag{10}$$

where $\theta \geq 0$ is the Kuhn-Tucker multiplier associated to the inequality $a' \geq 0$.

Using the envelope theorem

$$v'(a) = (1+r)u'(C), \tag{11}$$

and substituting (10) into (11) gives the well-known Euler equation:

$$u'(C) \geq \beta p(1+r)u'(C'), \text{ with equality if } a' > 0, \tag{12}$$

where C' is the consumption in the next period.

First-order-conditions: programming

The same consumption profile in (12) is given by using an algorithm that operates on (9). It involves the following steps:

1. Define a time-independent grid for assets, with $||a_{i+1} - a_i||$ sufficiently small,

$$G^a = \{a_1 = 0, a_2, a_3, \ldots, a_n\}, \tag{13}$$

where a_n is the maximum realization of assets weighted by units of effective labor.

a_1	`mina=0;`
a_n	`maxa=7;`
n	`Ngrid=70;`
G^a	`assets=mina:(maxa-mina)/(Ngrid-1):maxa;`
(a, a')	`[X,Y]=meshgrid(assets);`

```
[X,Y]=meshgrid(Assets);
```

which means

$$\underbrace{\begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ a_1 & a_2 & \ldots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \ldots & a_n \end{pmatrix}}_{\mathbf{X}} \quad \underbrace{\begin{pmatrix} a_1 & a_1 & \ldots & a_1 \\ a_2 & a_2 & \ldots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n & \ldots & a_n \end{pmatrix}}_{\mathbf{Y}}$$

Remark: *n is one of our most crucial parameters. When n is very high (let say 400) the accuracy of the results will be very good, however the program will take longer because it needs a lot of memory. A normal value for n will be in the range from 100 to 300 (but remember that there is always a tradeoff).*

2. Define the correspondence $f : G^a \to G^a$ of optimal combinations of assets at age x and $x+1$.

$$f(a_k) = a_j^* = \arg\max_{a_j} \left\{\lambda u(G(a_k, a_j) \geq 0) + \beta \cdot p \cdot v(a_j)\right\}, \quad \text{for any } a_k, a_j \in G^a. \tag{14}$$

Note that f has a one-to-one correspondence given that C is a convex set and $u(\cdot)$ is strictly concave.

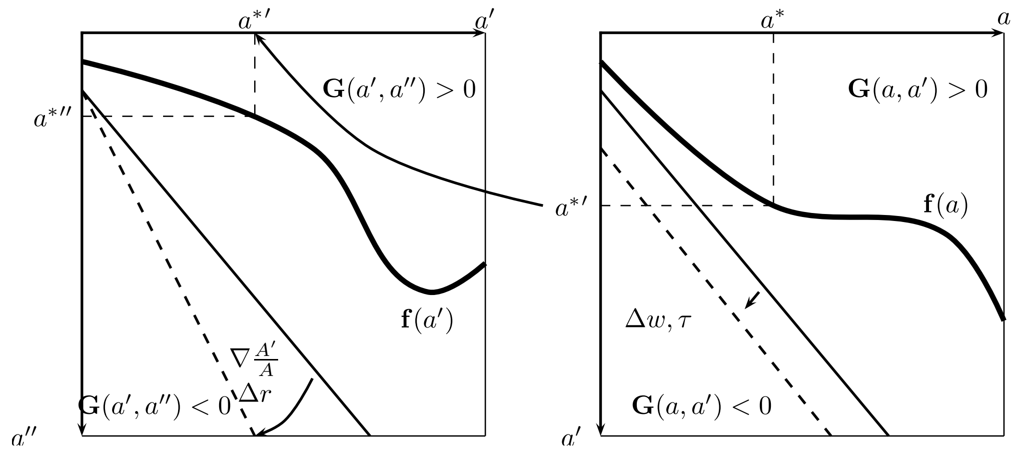

Figure 1: Optimal asset holdings during the life cycle using the Bellman Equation

3. Calculate the set $\{(a_k, f(a_k))\}_{k=1}^{n} \in C^{2n}$ of all possible optimal asset pairs for the household head each period.

```
for age=Omega:-1:1
Ce=(A(age)/UEC(age))*(r(age)*X-(A_g(age)*Y-X)+Yl(age)+PenB(age)+Bequests(age));
Je=((UEC(age)./(1-gamma)).*(Ce.^(1-gamma))+beta*Px(age)*(Ve(:,age+1)*ones(1,n)))...
.*(Ce>0).*(Ve(:,age+1)*ones(1,n)~=-1000); % Bellman equation
indNaNe=find(isnan(Je)); % find NaN data and substitute it for 0
Je(indNaNe)=0;
Je(find(Je==0))=-1000;
[utility_e arg_e]=max(Je,[],1); % mapping a_x-> a*_x+1
ind0_e=find(utility_e==-1000);
arg_e(ind0_e)=NaN;
Ve(:,age)=utility_e'; % Storage matrix, Expected Utility
ARG_e(:,age)=arg_e'; % Storage matrix, Position a_x-> a*_x+1
end
```

Remark: *By definition* `Ce` *must be greater than 0. Thus, we have excluded all combinations of present and future asset holdings such that the consumption is negative or equal to 0 (in the future* `Ve(:,age+1)*ones(1,n)~=-1000` *and in the present* `Ce>0`*).*

Remark: *It is important, at least for the case in which the individual can live for an infinite number of periods, that βp belongs to the interval $[0, 1)$, otherwise $v \to \infty$. In our case, this is guaranteed because the individual only lives for a finite number of periods, i.e. $p = 0$ for all $x \in \{\Omega, \Omega + 1, \dots\}$ and $\beta \in (0, 1]$.*

4. Use the initial boundary condition and plug it into $f(\cdot)$. Repeat this process for each period until the maximum lifespan of the household head is reached.

```
Opoint=find(X(1,:)==0);
Cap(1)=Opoint;
for age=1:Omega
Cap(age+1)=ARG_e(Cap(age),age);
end
```

3.2 Exercises:

Exercise 1. *Holding constant the real interest rate at $R = 3\%$, run one simulation with a 'grid' value of 0.1 and another one with a 'grid = .01'. How does the consumption profile change when the grid is smaller?*

Exercise 2. *Holding constant the real interest rate at $R = 5\%$, run one simulation with a 'maxa' value of 5 and another one with a 'maxa = 9'. Why does the consumption profile change when we increase 'maxa'?*

Exercise 3. *Holding constant the real interest rate at $R = 5\%$, run simulations for different ages of retirement 'T_r', replacement rates 'b', and payroll taxes 'τ'. Explain the results obtained.*

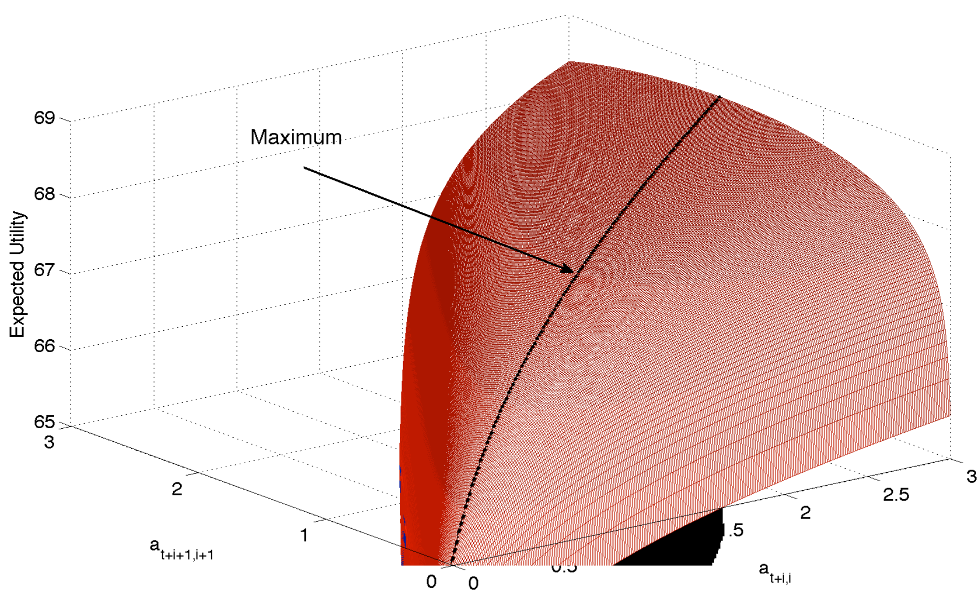

Figure 2: Expected utility values **v** and function f of an individual at age i in year $t + i$.

4 N Individuals: Bellman Equations

Since the general assumption is that the behavior of all economic agents is the same, an OLG model with N-individuals is modeled similarly than a model with only one representative individual. Hence, the new complexity lies on the fact that we need to add an extra dimension to our problem (matrices).

4.1 Overview of the programming strategy

The following diagram summarizes how the program seeks the equilibrium prices of the capital market. First, we introduce an initial real interest rate $\mathbf{R}^0$ into our program. Second, given that some transfers depend on interest rates and salaries, we will adjust transfers according to the new set of relative prices. Third, in order to obtain the asset holdings we will calculate the household's problem using the Bellman equation. Fourth, given the real interest rate we will sum asset holdings across age (total wealth) and we will divide it by the number of effective workers. Fifth, we will determine the interest rate ($\mathbf{R}^d$) at which the firm is willing to borrow that capital (in units of effective labor) from the households. Sixth, using the algorithm we will go through these steps again, if necessary.

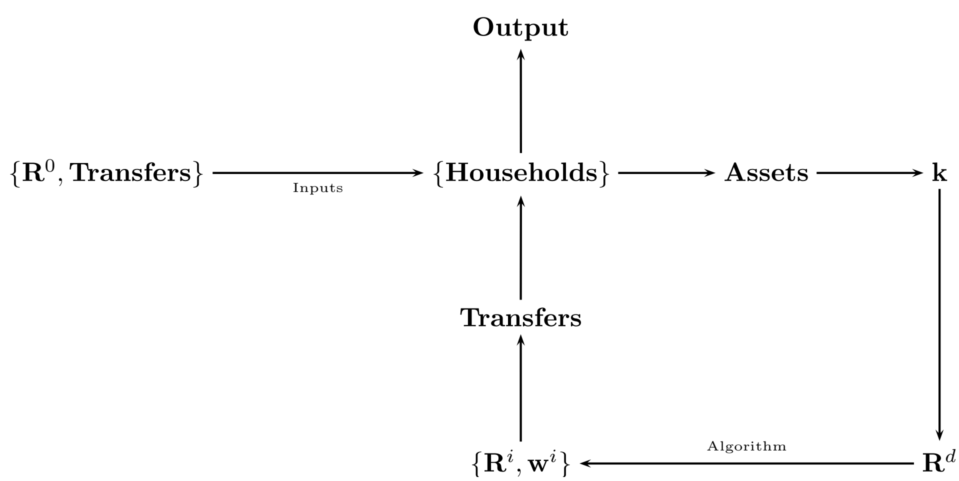

Figure 3: Diagram of the General Equilibrium OLG Model

4.2 Programming strategy: step by step

There are some peculiarities that distinguish an OLG model with respect to a representative individual problem. We introduce the differences in what follows

- i) **Final steady-state equilibrium**: we first need to determine what the expected utilities are for all the individuals alive in the last period. To do this, we will assume a representative individual and we will calculate the expected utility at each age (Note that this part is not an overlapping generations model).

```
for age=Omega-1:-1:u
C=(1+R(T))*X-Agr(T).*Y+E(T,age).*W(T)+Bequest(T,age);
Utility=(log(A(T).*C)+beta.*px(T,age).*(Ve(age+1,:)'*ones(1,length(Y)))).*...
(C>0).*(Ve(age+1,:)'*ones(1,length(Y))~=-1000);
Utility(find(isnan(Utility)))=0;
Utility(find(Utility==0))=-1000;
[Ve(age,:)  Fe(age,:)]=max(Utility,[],1);
Fe(find(Ve==-1000))=NaN;
end
```

- ii) **Timing and aggregation problems**: The goal of the algorithm is to find the equilibrium price in the capital market. This is always satisfied unless the demand and supply of capital does not cross, which in my experience is extremely rare. For this reason, we have to check additional equations, or impose additional conditions, in order to guarantee that the algorithm finds the correct solution. The best candidate is the market clearence condition. We should obtain the market clearance condition by summing the flow budget constraints across age.

$$\sum_{x=0}^{\Omega-1} c_{t,x} N_{t+1,x+1} = \sum_{x=0}^{\Omega-1} ((1+r_t)(a_{t,x}+h_{t,x}) - a_{t+1,x+1} + \epsilon_x w_t A_t + \tau_{t,x}) N_{t+1,x+1} \tag{15}$$

where

$$C_t = \sum_{x=0}^{\Omega-1} c_{t,x} N_{t+1,x+1} \tag{16}$$

$$K_t = \sum_{x=0}^{\Omega-1} (a_{t,x}+h_{t,x}) N_{t+1,x+1} \tag{17}$$

$$K_{t+1} = \sum_{x=0}^{\Omega-1} a_{t+1,x+1} N_{t+1,x+1} \tag{18}$$

$$L_t = \sum_{x=0}^{\Omega-1} \epsilon_x N_{t+1,x+1} \tag{19}$$

$$0 = \sum_{x=0}^{\Omega-1} \tau_{t,x} N_{t+1,x+1} \tag{20}$$

Using equations (15)-(20) we have that

$$C_t + \underbrace{K_{t+1} - K_t}_{S_t} = r_t K_t + w_t A_t L_t \tag{21}$$

- iii) **Dealing with N simultaneous Bellman equations**: Now, we have several ages to compute, therefore we have to incorporate an additional dimension in our matrices and maximize across ages. The result of this process will be the supply of capital.

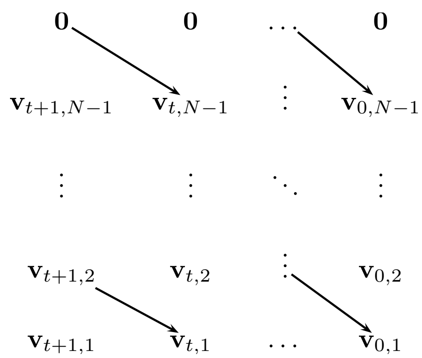

Figure 4: N Bellman equations in year t

```
for time=T-1:-1:1 ABR=reshape(kron(ones(1,Omega-1),(1+R(time))*X-Y*Agr(time)),[L L
Omega-1]); % Asset Based Reallocation in units of effective labor
C=ABR+W(time)*reshape(kron(E(time,:),ones(L)),[L L Omega-1])...
+reshape(kron(Bequest(time,:),ones(L)),[L L Omega-1]); % Consumption of all
individuals in year 'time'
EU=(log(A(time)*C)+beta*reshape(kron(px(time,:),ones(L)),[L L Omega-1])...
.*reshape(kron(V(:,u+1:Omega,time+1),ones(1,L)),[L L Omega-1])).*(C>0)...
.*(reshape(kron(V(:,u+1:Omega,time+1),ones(1,L)),[L L Omega-1])~=-1000);
EU(find(isnan(EU)))=0;
EU(find(EU==0))=-1000;
[J ArgJ]=max(EU,[],1);
% Bellman Equation Vt,
V(:,u:Omega-1,time)=reshape(J,[L Omega-1]);
% Argument at,x->at+1,x+1
F(u:Omega-1,:,time)=reshape(ArgJ,[L Omega-1])';
F(find(V==-1000))=NaN;
end
```

From these equations and using the initial condition $\tilde{a}_{t,0} = 0, \forall t$, we will derive the optimal asset holdings for all individuals over time $\{\tilde{a}_{t,x}\}_{x=0}^{\Omega}$ for all $t \in \{0, \dots, T\}$.

```
IniAssets=ones(T,Omega);
for age=u:Omega-1
for time=1:T-2
IniAssets(time+1,age+1)=F(age,IniAssets(time,age),time);
end
end
```

- iv) **Demand for Capital**: from step (iii) we can calculate the stock of capital in units of effective labor,

$$k_t = \frac{\sum_{x=0}^{\Omega-1} \tilde{a}_{t,x} N_{t,x}}{L_t}. \tag{22}$$

```
assets(1:T-1,:)=[Assets(IniAssets(1:T-1,:))];
Kapital=sum(assets(1:T-1,u:Omega-1).*Pop(1:T-1,u:Omega-1),2)./...
sum(Pop(2:T,u:Tr-1),2);
```

Now, using the Cobb-Douglas production function, we can calculate the interest rate (r_t^d) that the firm is willing to pay in order to use k_t.

- v) **Algorithm**: In order to find the equilibrium interest rate we will use a fixed point algorithm (Fair, 1996):

$$r_t^i = r_t^{i-1} + \xi({r^d}_t - r_t^{i-1}), \text{ for all } t \in \{0, \ldots, T\} \tag{23}$$

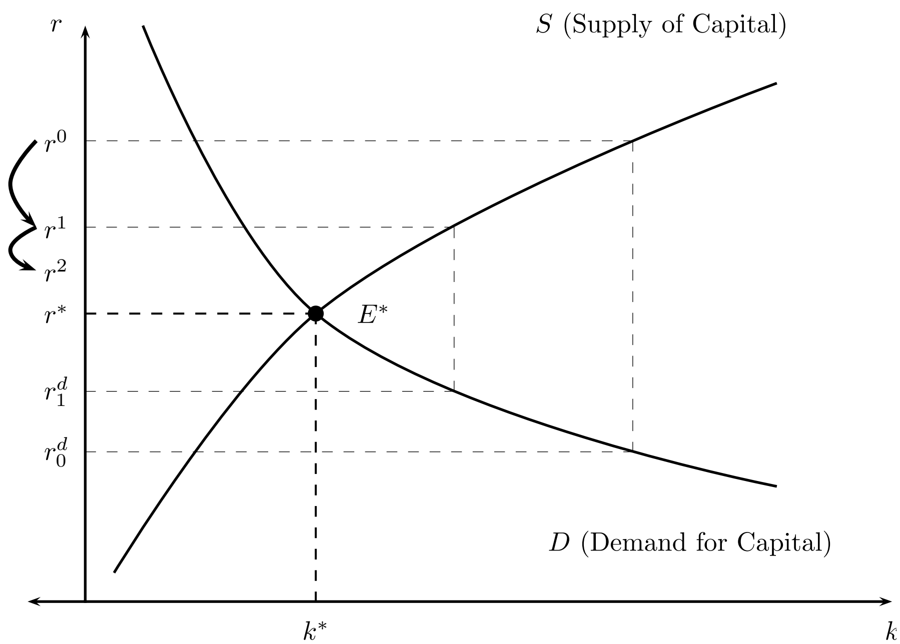

Figure 1: Fixed Point Algorithm: $\mathbf{r}^i = \mathbf{r}^{i-1} + \xi(\mathbf{r}^d - \mathbf{r}^{i-1})$

The salary is derived using the Cobb-Douglas production function as follows:

$$F(K_t, A_tL_t) = (K_t)^{\alpha}(A_tL_t)^{1-\alpha} \Rightarrow \begin{cases} r_t = \alpha\left(\frac{K_t}{A_tL_t}\right)^{\alpha-1} - \delta \\ \\ w_t = \alpha\left(\frac{K_t}{A_tL_t}\right)^{\alpha} \end{cases} \Rightarrow w_t = \alpha\left(\frac{r_t+\delta}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \tag{24}$$

5 Data requirements

This section is devoted to introduce the information needed to run your own simulation. The information is divided into two main sets: demographic data and economic data.

5.1 Demographic data

National Transfer Accounts database contain economic information by single years of age. To properly use this information it is necessary to model the past, present, and future of the population. This task requires three sets of demographic data: age-specific mortality rates, age-specific fertility rates, and population by age and time. Except for the population by age and time, the simulation model will not use this information directly. However, the first two sets are used to estimate and forecast survival probabilities, number of children within the household, and (if necessary) net migration rates over time.

When modeling the population remember that we have to satisfy the following assumptions:

- **There is no sex difference**: all demographic data must combine males and females, i.e. total.
- **Fertility must start when the individual makes decisions**: fecundity has to be truncated.
- **Initial and final populations must be stable**: leave age-specific mortality rates and age-specific fertility rates unchanged outside of the period to be analyzed. As a rule of thumb both initial and final stable populations could last 200 years each.

5.2 Economic data

General equilibrium OLG models need time series of micro- and macroeconomic data. Some information can be found directly in the NTA database (e.g. units of equivalent adult consumers, units of effective labor), whereas the rest of the information will be taken from national and international institutions (e.g. UN, World Bank, OCDE, etc...). The variables that we should have information on are the following:

- Time series of National Accounts (real terms)
  From NA we calculate: i) the stock of capital using the time series of "Gross Fixed Capital Formation" and the "Consumption of Fixed Capital", ii) the average rate of depreciation of capital (δ), and iii) the capital share (α).
- Age-specific labor productivity indexes (cross-sectional or time series)
- Labor force participation rates (cross-sectional or time series)
- Labor productivity growth rates (time series)

$$\frac{\Delta A_t}{A_t} \approx \frac{1}{1-\alpha}\frac{\Delta y_t}{y_t} - \frac{\alpha}{1-\alpha}\frac{\Delta k_t}{k_t} - \frac{\Delta (L/N)_t}{(L/N)_t}, \tag{25}$$

  where α is the capital share, y is the GDP per capita, and k is the stock of capital per capita.

- Transfers (NTA)

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